SEMI-REGULAR PLANE POLYGONS OF INTEGRAL TYPE

BY

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ABSTRACT

A polygon is called semi-regular if its interior angles are equal to one another. The paper deals mainly with semi-regular polygons whose sides have integral lengths.

A plane polygon of order n is called *semi-regular* if its vertex angles have the common value $(n-2)\pi/n$ and is said to be of *integral type* if each of its edges is in length an integer multiple of a given unit length. In this paper we characterize these polygons in such a way that we can answer two kinds of questions. First we show that there exists a semi-regular polygon of order $n > 1$ whose edge set consists of any n consecutive integers if and only if n is not a power of a prime. Secondly, for selected values of n, such as $n = p^\alpha$, $n = p^\alpha q^\beta$, where p and q are distinct primes, we determine $S(n,k)$, the number of classes of semi-regular polygons of integral type of order n, having each side $a(i)$ in the range $1 \leqq a(i) \leqq k$, and with the classes distinct under the permutations of the dihedral group G of order $|G| = 2n$.

Part 1. Characterization theorems and corollaries

Using the vector representation for the sum and product of complex numbers, we may give an algebraic description of a semi-regular polygon of order n whose edges are, in order, a suitable set of positive real numbers $a(0), a(1), \cdots, a(n-1)$ by insisting that $\eta_0 = e^{2\pi i/n}$ be a solution of $P(z) = 0$ for the polynomial

$$P(z) = \sum_{\nu=0}^{n-1} a(\nu) z^\nu.$$

Let us understand that

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(1) $$a(\mu) = a(\nu) \text{ when } \mu \equiv \nu \bmod n.$$

LEMMA 1. *If (1) holds and if η is any n-th root of unity, then for any integer c,*

(2) $$\sum_{\nu=0}^{n-1} a(\nu)\eta^{\nu+c} = \sum_{\nu=0}^{n-1} a(\nu - c)\eta^{\nu}.$$

PROOF. Since $\eta^{\mu} = \eta^{\nu}$ when $\mu \equiv \nu \bmod n$ and since (1) holds, the change in the index of summation is readily justified.

THEOREM 1. *Let p be any prime, α any positive integer, and $P(z) = \sum_{\nu=0}^{p^{\alpha}-1} a(\nu)z^{\nu}$ a polynomial with rational coefficients. Let $\eta_0 = e^{2\pi i/p^{\alpha}}$. Then $P(\eta_0) = 0$ if and only if*

(3) $$a(\nu) - a(\nu + p^{\alpha-1}) = 0$$

for all integers ν, where it is understood that (1) holds with $n = p^{\alpha}$.

NECESSITY. Assume $P(\eta_0) = 0$. Since the coefficients of $P(z)$ are rational and since η_0 is a primitive p^{α}-th root of unity, $P(z)$ is divisible by $\Phi_{p^{\alpha}}(z)$, the cyclotomic polynomial, so $P(\varepsilon) = 0$ for all primitive p^{α}-th roots of unity ε. Imprimitive p^{α}-th roots of unity satisfy $z^{p^{\alpha-1}} - 1 = 0$, hence every p^{α}-th root of unity η, primitive or not, satisfies $Q(z) = 0$, where

(4) $$Q(z) = (z^{p^{\alpha-1}} - 1)P(z).$$

Using (2) we may rewrite

(5) $$0 = Q(\eta) = \sum_{\nu=0}^{p^{\alpha}-1} a(\nu)\,(\eta^{\nu+p^{\alpha-1}} - \eta^{\nu})$$

in the form

(6) $$\sum_{\nu=0}^{p^{\alpha}-1} [a(\nu - p^{\alpha-1}) - a(\nu)]\eta^{\nu} = 0.$$

But such a polynomial of degree $p^{\alpha} - 1$, with p^{α} roots, must have every coefficient vanish. Hence

(7) $$a(\nu - p^{\alpha-1}) - a(\nu) = 0, \qquad 0 \leqq \nu \leqq p^{\alpha} - 1.$$

This is equivalent to (3) if we set $\nu' = \nu - p^{\alpha-1}$.

SUFFICIENCY. Assume (3). Rewrite (3) as (7) which implies (6) for η_0. By (2) rewrite (6) as (5). Then use (4) so that

(8) $$(\eta_0^{p\alpha-1} - 1)P(\eta_0) = 0.$$

But η_0 is a primitive p^α-th root of unity, so that $\eta_0^{p^{\alpha-1}} - 1 \neq 0$. Hence $P(\eta_0) = 0$, as desired.

The significance of (3) is that for a semi-regular polygon of order p^α, exactly $p^{\alpha-1}$ parameters may be assigned, say $a(1), a(2), \cdots, a(p^{\alpha-1})$, and then the other $a(i)$ are determined in "p-gon sets" by

$$(9) \quad a(s) = a(s + p^{\alpha-1}) = a(s + 2p^{\alpha-1}) = \cdots = a(s + (p-1)p^{\alpha-1}), \qquad 1 \leqq s \leqq p^{\alpha-1}.$$

COROLLARY 1. *There is no semi-regular polygon of order $n = p^\alpha$ whose edge set consists of n consecutive integers.*

THEOREM 2. *Let p and q be primes, $p \neq q$, and let $P(z) = \sum_{v=0}^{pq-1} a(v) z^v$ be a polynomial with rational coefficients. Let $\eta_0 = e^{2\pi i/pq}$. Then $P(\eta_0) = 0$ if and only if*

$$(3') \qquad a(v) - a(v+p) - a(v+q) + a(v+p+q) = 0$$

for all integers v, where it is understood that (1) *holds with $n = pq$.*

NECESSITY. Assume $P(\eta_0) = 0$. Since the coefficients of $P(z)$ are rational and since η_0 is a primitive pq-th root of unity, $P(z)$ is divisible by the cyclotomic polynomial $\Phi_{pq}(z)$, so $P(\varepsilon) = 0$ for all primitive pq-th roots of unity ε. Imprimitive pq-th roots of unity satisfy either $z^p - 1 = 0$ or $z^q - 1 = 0$, hence every pq-th root of unity η, primitive or not, satisfies $Q(z) = 0$, where

$$(4') \qquad Q(z) = (z^p - 1)(z^q - 1)P(z).$$

Using (2) we may rewrite

$$(5') \qquad 0 = Q(\eta) = \sum_{v=0}^{pq-1} a(v)(\eta^{v+p+q} - \eta^{v+p} - \eta^{v+q} + \eta^v)$$

in the form

$$(6') \qquad \sum_{v=0}^{pq-1} [a(v-p-q) - a(v-p) - a(v-q) + a(v)]\eta^v = 0.$$

But such a polynomial of degree $pq - 1$, with pq roots, must have every coefficient vanish. Hence

$$(7') \qquad a(v-p-q) - a(v-p) - a(v-q) + a(v) = 0, \qquad 0 \leqq v \leqq pq - 1.$$

This is equivalent to (3′) if we set $v' = v - p - q$.

SUFFICIENT. Assume (3′). Rewrite (3′) as (7′) which implies (6′) for η_0. By (2) rewrite (6′) as (5′). Then use (4′) so that

(8′) $$(\eta_0^p - 1)(\eta_0^q - 1)P(\eta_0) = 0.$$

But η_0 is a primitive pq-th root of unity, so $\eta_0^p - 1 \neq 0$ and $\eta_0^q - 1 \neq 0$. Hence $P(\eta_0) = 0$, as desired.

COROLLARY 2. *There exists a semi-regular polygon of order $n = pq$ whose edge set is exactly $\{a(s)\} = \{1,2,\cdots,n\}$.*

PROOF. It is well known from number theory that given $(u,v) = 1$, the set $\{xu + yv\}$ represents a complete residue system mod uv, if x and y run independently through complete residue systems mod v and mod u, respectively.

Take $u = p$ and $v = q$, distinct primes, so $(p,q) = 1$. Let x run through the values $0, 1,\cdots, q-1$, a complete residue system mod q; let y run through the values $1,2,\cdots,p$, a complete residue system mod p. Then $\{xp + yq\}$ represents a complete residue system mod pq.

Under the above conditions on x and y, the set $\{xp + y\}$ represents the integers $\{1,2,\cdots,pq\}$; for if $1 \leqq a \leqq pq$ we can modify the division algorithm to write $a = xp + y$, $1 \leqq y \leqq p$, and because of the limits on a, we have $0 \leqq x \leqq q - 1$.

Thus the correspondence $[xp + yq]T = xp + y$ is a one-to-one mapping from the residue classes mod pq onto the set of positive integers from 1 to pq. If $s \equiv xp + yq$ mod pq, we let $a(s) = xp + y$.

EXAMPLE. $p = 3$, $q = 5$.

x	0	0	0	1	1	1	2	2	2	3	3	3	4	4	4
y	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
$a(s)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$xp + yq$	5	10	15	8	13	18	11	16	21	14	19	24	17	22	27
s	5	10	15	8	13	3	11	1	6	14	4	9	2	7	12

Using the correspondence $[s]T = a(s)$, we can show for every residue class $[s]$ mod pq that

(3′) $$a(s) + a(s + p + q) = a(s + p) + a(s + q).$$

If $0 \leqq x < q - 1$, $1 \leqq y < p$, then for

$$\begin{aligned} s &\equiv xp + yq, & a(s) &= xp + y; \\ s + p + q &\equiv (x+1)p + (y+1)q, & a(s + p + q) &= (x+1)p + (y+1); \\ s + p &\equiv (x+1)p + yq, & a(s + p) &= (x+1)p + y; \\ s + q &\equiv xp + (y+1)q, & a(s + q) &= xp + (y+1). \end{aligned}$$

If $0 \leqq x < q-1$, $y = p$, then for

$$\begin{aligned} s &\equiv xp + pq, & a(s) &= xp + p;\\ s+p+q &\equiv (x+1)p + q, & a(s+p+q) &= (x+1)p + 1;\\ s+p &\equiv (x+1)p + pq, & a(s+p) &= (x+1)p + p;\\ s+q &\equiv xp + q, & a(s+q) &= xp + 1. \end{aligned}$$

If $x = q-1$, $1 \leqq y < p$, then for

$$\begin{aligned} s &\equiv (q-1)p + yq, & a(s) &= (q-1)p + y;\\ s+p+q &\equiv (y+1)q, & a(s+p+q) &= y + 1;\\ s+p &\equiv yq, & a(s+p) &= y;\\ s+q &\equiv (q-1)p + (y+1)q, & a(s+q) &= (q-1)p + (y+1). \end{aligned}$$

If $x = q-1$, $y = p$, then for

$$\begin{aligned} s &\equiv (q-1)p + pq, & a(s) &= (q-1)p + p;\\ s+p+q &\equiv q, & a(s+p+q) &= 1;\\ s+p &\equiv pq, & a(s+p) &= p;\\ s+q &\equiv (q-1)p + q, & a(s+q) &= (q-1)p + 1. \end{aligned}$$

Thus in all four cases (3′) holds.

The proof of Corollary 2 is now complete, by reference to the sufficiency part of Theorem 2.

COROLLARY 3. *If p and q are distinct primes, then for any positive integer k, there exists a semi-regular polygon of order $n' = kpq$ whose edge set is exactly $\{a(s)\} = \{1, 2, \cdots, n'\}$.*

PROOF. Given a semi-regular polygon of integral type of order n, represented by $\sum_{j=0}^{n-1} a(j)\,\eta_0^j = 0$, where $\eta_0 = e^{2\pi i/n}$, we may use $M\sum_{j=0}^{n-1}\eta_0^j = 0$, corresponding to a regular polygon of order n and of side M, to obtain $\sum_{j=0}^{n-1}[a(j)+M]\,\eta_0^j = 0$, representing a semi-regular polygon, related to the original one, by having every side increased by the same amount M. For any positive integer k we have

$$\sum_{r=0}^{k-1} \eta_0^{r/k}\left(\sum_{j=0}^{n-1}[a(j)+M_r]\eta_0^j\right) = 0.$$

If we set $\mu_0 = \eta_0^{1/k}$, then $\eta_0^{r/k}\eta_0^j = \mu_0^{r+kj}$. If we set $b(r+kj) = a(j) + M_r$, the last displayed equation becomes

$$\sum_{m=0}^{nk-1} b(m)\mu_0^m = 0.$$

Thus for integral values of $M_0, M_1, \cdots, M_{k-1}$, we have obtained a semi-regular polygon of order $n' = nk$ with edge set $\{b(m)\} = \{a(j) + M_r\}$.

From Corollary 2, when $n = pq$, we may suppose $\{a(j)\} = \{1, 2, \cdots, n\}$. If we take $M_r = rn$, we obtain $\{b(m)\} = \{1, 2, \cdots, kn = n'\}$, as desired.

For example, using the previous result for $n = 15$:

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$a(j)$	3	8	13	6	11	1	9	14	4	12	2	7	15	5	10

we obtain a solution for $n' = 30 = 2n$, as follows:

$r = 0$	m	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28
$M_0 = 0$	$b(m)$	3	8	13	6	11	1	9	14	4	12	2	7	15	5	10
$r = 1$	m	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29
$M_1 = 15$	$b(m)$	18	23	28	21	26	16	24	29	19	27	17	22	30	20	25

THEOREM 3. *There exists a semi-regular polygon of order $n > 1$ whose edge set is exactly $\{a(j)\} = \{1, 2, \cdots, n\}$ if and only if $n \neq p^\alpha$.*

PROOF. Combine Corollary 1 and Corollary 3.

We deliberately used the same pattern of proof for Theorem 1 and Theorem 2 to make it clear that the method used to establish (3) and (3′) can be extended to establish similar conditions for any integer n. A multiplier $G(z)$, like those used in (4) and (4′), can be found for the general case, by examining the formula for the cyclotomic polynomial:

$$\Phi_n(z) = \prod_{d|n} (z^d - 1)^{\mu(n/d)};$$

and selecting $G(z) = \prod' (z^d - 1)$, where the product contains just those terms for which n/d is a prime.

For example, if $n = p^\alpha q^\beta$, where p and q are distinct primes, then

$$G(z) = (z^{p^{\alpha-1}q^\beta} - 1)(z^{p^\alpha q^{\beta-1}} - 1);$$

so the condition analogous to (3) and (3′) is the following:

$$(3'') \qquad a(v) - a(v + p^{\alpha-1}q^\beta) - a(v + p^\alpha q^{\beta-1}) + a(v + p^{\alpha-1}q^\beta + p^\alpha q^{\beta-1}) = 0,$$

for every integer v.

Part 2. Counting problems

For any positive integer k, let $T(n,k)$ be the number of ways of assigning integers $a(1)$, $a(2),\cdots,a(n)$ with

$$(10) \qquad 1 \leqq a(v) \leqq k, \qquad v = 1,2,\cdots,n;$$

to produce a semi-regular polygon with, say, $\Sigma_{v=1}^{n} a(v)\eta_0^v = 0$ for $\eta_0 = e^{2\pi i/n}$.

Two of these semi-regular polygons counted by $T(n,k)$ are said to belong to the same equivalence class if one can be obtained from the other one by a permutation of the dihedral group $G = \{g\}$ of order $|G| = 2n$. Let $S(n,k)$ denote the number of equivalence classes.

Except for the case $n = p^\alpha$, we find the determination of $T(n,k)$ to be difficult. Some of our ad hoc methods appear in the following sections. But once $T(n,k)$ is known, there is a standard method for finding $S(n,k)$, using the Frobenius-Burnside formula (see [2])

$$(11) \qquad S(n,k) = \frac{1}{|G|} \sum_{g \in G} N(g),$$

where $N(g)$ is the number of polygons in the set counted by $T(n,k)$ which are left invariant by g. It is easy to classify the "rotations" of G according to cyclic type, there being exactly $\phi(d)$ of primitive period d, for each divisor d of n. The "reflections" of G are all of one type, if n is odd; but there are $n/2$ of one type and $n/2$ of another type, if n is even. Thus the application of (11) is reasonably straightforward, but depends on the factorization of n and the corresponding simplicity or complexity of the conditions like (3) or (3′). If $N(I) = T(n,k)$ can be found for the identity permutation I, it seems that $N(g)$ for any other g in G is readily found.

The Case $n = p^\alpha$. Since the conditions (9) are necessary and sufficient and since there are exactly k values for each of the independent parameters $a(s)$ for $1 \leqq s \leqq p^{\alpha-1}$, we find $T(p^\alpha, k) = k^{p^{\alpha-1}}$.

Let R be a rotation of G of period $p^{\alpha-t}$ for t in the range $0 \leqq t \leqq \alpha - 1$. In order for the polygon with edge set $a(1)$, $a(2),\cdots,a(n)$ to be invariant under R, since $p^\alpha/p^{\alpha-t} = p^t$, the parameters $a(s)$ for $1 \leqq s \leqq p^{\alpha-1}$ must satisfy the condition $a(v) = a(v + p^t)$, hence $p^{\alpha-1}/p^t$ of them are equal. Thus p^t is the number of independent ones, and since each of these has k possible values, $N(R) = k^{p^t}$. For later use we note there are $\phi\ (p^{\alpha-t}) = (p-1)p^{\alpha-t-1}$ rotations with the same period as R.

Assume p is odd. Let F be a reflection of G. Since we may assume any $p^{\alpha-1}$ consecutive sides as the independent parameters in (9), it is not too special to assume F interchanges $a(i)$ and $a(p^\alpha - i + 1)$ for $1 \leqq i \leqq (p^\alpha - 1)/2$, but fixes $a(j)$ for $j = (p^\alpha + 1)/2$. Since a semi-regular polygon with sides $a(1), a(2), \cdots, a(n)$ satisfying (9) has its subscripts reducible modulo $p^{\alpha-1}$, such a polygon is invariant under F if and only if the independent parameters $a(s)$ for $1 \leqq s \leqq p^{\alpha-1}$ satisfy the additional conditions $a(1) = a(p^{\alpha-1}), \cdots, a((p^{\alpha-1} - 1)/2) = a((p^{\alpha-1} + 3)/2)$, but with no restriction on $a((p^{\alpha-1} + 1)/2)$. Thus the number of independent parameters for a polygon left invariant by F is given by $(p^{\alpha-1} - 1)/2 + 1 = (p^{\alpha-1} + 1)/2$. Hence for each of the p^α reflections F in G, $N(F) = k^{(p^{\alpha-1}+1)/2}$.

Remembering that $N(I) = T(p^\alpha, k)$, we combine the results above with (11) to find, for p odd,

$$S(p^\alpha, k) = \frac{1}{2p^\alpha}\left[k^{p^{\alpha-1}} + \sum_{t=0}^{\alpha-1} (p-1)p^{\alpha-t-1}k^{p^t} + p^\alpha k^{(p^{\alpha-1}+1)/2}\right].$$

When $\alpha = 1$, since there is only one type of semi-regular p-gon, namely, the regular p-gon, we anticipate and check that $T(p,k) = k$ and $S(p,k) = k$, for $S(p,k) = (1/2\,p)[k + (p-1)k + pk] = k$. If $\alpha = 2$, we find

$$\begin{aligned} S(p^2, k) &= \frac{1}{2p^2}[k^p + (p-1)k^p + (p-1)pk + p^2k^{(p+1)/2}] \\ &= \frac{k}{2p}[k^{(p-1)/2} + 1][k^{(p-1)/2} + (p-1)]. \end{aligned}$$

In particular,

$$S(9,k) = \frac{k}{6}(k+1)(k+2) = \binom{k+2}{3};$$

$$S(25,k) = \frac{k}{10}(k^2+1)(k^2+4).$$

Both of these formulas can be derived directly by a combinatorial argument. As an illustration, we check $S(25,3) = 39$, by describing a representative for each of the 39 classes; for such a representative, we need name only $a(1)$, $a(2)$, $a(3)$, $a(4)$, $a(5)$, since (9) prescribes the remaining sides:

$a(1)$	1	2	3	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2
$a(2)$	1	2	3	1	1	1	2	2	2	1	1	1	3	3	3	2	2	2	3	3	3
$a(3)$	1	2	3	1	1	2	2	1	1	1	1	3	3	1	1	2	2	3	3	2	2
$a(4)$	1	2	3	1	2	2	2	1	2	1	3	3	3	1	3	2	3	3	3	2	3
$a(5)$	1	2	3	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3

$a(1)$	1	1	2	2	3	3	1	1	1	1	2	2	2	2	3	3	3	3
$a(2)$	1	1	2	2	3	3	2	2	2	3	1	1	1	3	1	1	1	2
$a(3)$	1	2	2	1	3	1	2	3	3	2	1	3	3	1	1	2	2	1
$a(4)$	2	1	1	2	1	3	3	3	2	2	3	3	1	1	2	2	1	1
$a(5)$	3	3	3	3	2	2	3	2	3	3	3	1	3	3	2	1	2	2

Of course, these representatives must be chosen so that no two are equivalent under rotation or reflection.

Assume $p = 2$, $\alpha > 1$. Let D be a reflection of G which leaves no edge fixed. We may assume the notation chosen so that D interchanges $a(i)$ and $a(2^\alpha - i + 1)$ for $1 \leqq i \leqq 2^{\alpha-1}$. For a polygon satisfying (9), the subscripts are reducible modulo $2^{\alpha-1}$, so such a polygon is invariant under D if and only if the parameters $a(1)$, $a(2), \cdots, a(2^{\alpha-1})$ satisfy the additional conditions $a(1) = a(2^{\alpha-1}), \cdots, a(2^{\alpha-2}) = a(2^{\alpha-2}+1)$. Thus the number of independent parameters for a polygon left invariant by D is given by $2^{\alpha-2}$, so $N(D) = k^{2^{\alpha-2}}$. There are $2^{\alpha-1}$ reflections like D.

Let M be a reflection of G which leaves two edges fixed. Assume the notation chosen so that M fixes $a(1)$ and $a(2^{\alpha-1}+1)$ and interchanges $a(i)$ and $a(2^\alpha - i + 2)$ for $2 \leqq i \leqq 2^{\alpha-1}$. For a polygon satisfying (9), the subscripts are reducible modulo $2^{\alpha-1}$, so such a polygon is invariant under M if and only if the parameters $a(1)$, $a(2), \cdots, a(2^{\alpha-1})$ satisfy the additional conditions $a(2) = a(2^{\alpha-1})$, $a(3) = a(2^{\alpha-1}-1)$, $\cdots, a(2^{\alpha-2}) = a(2^{\alpha-2}+2)$, leaving both $a(1)$ and $a(2^{\alpha-2}+1)$ unrestricted. Thus the number of independent parameters for a polygon left invariant by M is given by $(2^{\alpha-2} - 1) + 2 = 2^{\alpha-2} + 1$, so $N(M) = k^{2^{\alpha-2}+1}$. There are $2^{\alpha-1}$ reflections like M.

Remembering that $N(I) = T(2^\alpha, k)$, we combine the previous results on rotations and the above results on reflections with (11) to find, for p even,

$$S(2^\alpha, k) = \frac{1}{2^{\alpha+1}} \left[k^{2^{\alpha-1}} + \sum_{t=0}^{\alpha-1} 2^{\alpha-1-t} k^{2^t} + 2^{\alpha-1} k^{2^{\alpha-2}}(1+k) \right].$$

In particular $S(4,k) = (1/8)[k^2 + k^2 + 2k + 2k(1+k)] = k(k+1)/2 = \binom{k+1}{2}$; which is easy to check directly, since a semi-regular polygon for $n = 4$ is a rectangle or a square, so $S(4,k) = k(k-1)/2 + k$. For another example, after simplification, we find $S(8,k) = k(k+1)(k^2+k+2)/8$. It is easy to check that for $\alpha \geqq 2$, the numerator in $S(2^\alpha, k)$ has the factor $k(k+1)$.

The Case $n = pq,\ p \neq q$. We replace the "standard" conditions

$$(3') \qquad a(v) - a(v+p) - a(v+q) + a(v+p+q) = 0, \qquad 0 \leqq v \leqq n-1,$$

by an equivalent set

$$(9') \qquad a(yp+xq) = C_y - A_x, \qquad 1 \leqq x \leqq p-1,\ 0 \leqq y \leqq q-1,$$

where we use the remarks in Corollary 2 to express $v \equiv yp + xq \bmod pq$ uniquely in terms of x and y in the ranges $0 \leqq x \leqq p-1$, $0 \leqq y \leqq q-1$, and then we define

$$C_y = a(yp), \qquad 0 \leqq y \leqq q-1,$$
$$A_x = a(0) - a(xq), \qquad 1 \leqq x \leqq p-1.$$

To show that (3′) implies (9′), we iterate (3′) replacing v by $v+q$:

$$a(v) - a(v+p) = a(v+q) - a(v+q+p) = \cdots = a(v+xq) - a(v+xq+p).$$

But this may be written as follows and then iterated, replacing v by $v+p$:

$$a(v) - a(v+xq) = a(v+p) - a(v+xq+p) = \cdots = a(v+yp) - a(v+xq+yp).$$

Setting $v = 0$, we obtain $a(0) - a(xq) = a(yp) - a(xq+yp)$ which is equivalent to (9′) using the definitions of C_y and A_x.

Conversely, to show that (9′) implies (3′) we have $a(v) - a(v+p) - a(v+q) + a(v+p+q) = (C_y - A_x) - (C_{y+1} - A_x) - (C_y - A_{x+1}) + (C_{y+1} - A_{x+1}) = 0$.

The significance of (9′) is that every edge $a(v)$ can be expressed in terms of the set of values $C_y = a(yp)$, $0 \leqq y \leqq q-1$, and the $p-1$ auxiliary parameters $A_1, A_2, \cdots, A_{p-1}$.

To find $T(pq,k)$ we study the effect of adding the restriction (10) to the condition (9′). The requirements $1 \leqq a(i) \leqq k$ imply that the A_x in (9′) must satisfy the condition $|A_x| \leqq k-1$. We suppose that $A_1, A_2, \cdots, A_{p-1}$ have been specified with $|A_x| \leqq k-1$.

Case 1. Suppose $A_x \geqq 0$. Then (9′) shows $a(xq) = a(0) - A_x$, which together with (10) shows the choices of $a(0)$ limited to $A_x + 1, \cdots, k$, a total of $k - A_x$ choices. Let $A^+ = \{A_x, A_x \geqq 0\}$. If $A^+ \neq \varnothing$, let $A_i^* = \max A_x$ over the A_x in A^+. (If $A^+ = \varnothing$, define $A_i^* = 0$.) The greatest restriction is placed on $a(0)$ by A_i^*. So, from the point of view of Case 1, there are exactly $k - A_i^*$ choices for $a(0)$.

Case 2. Suppose $A_x < 0$. Then (9′) shows $a(xq) = a(0) + |A_x|$, which together with (10) shows the choices of $a(0)$ limited to $1, 2, \cdots, k - |A_x|$, a total of $k - |A_x|$ choices. Let $A^- = \{A_x, A_x < 0\}$. If $A^- \neq \emptyset$, let $A_j^* = \max|A_x|$ over the A_x in A^-. (If $A^- = \emptyset$, define $A_j^* = 0$.) The greatest restriction is placed on $a(0)$ by $-A_j^*$. So, from the point of view of Case 2, there are exactly $k - A_j^*$ choices for $a(0)$.

However, we must consider that both Case 1 and Case 2 may hold for the set $A_1, A_2, \cdots, A_{p-1}$ being studied. When we compare $A_i^* + 1, \cdots, k$, the least range of $a(0)$ for all A_x in A^+, and $1, 2, \cdots, k - A_j^*$, the least range of $a(0)$ for all A_x in A^-, we see there is either no common range, when $A_i^* + A_j^* \geqq k$, or the number of choices is given by $(k - A_j^*) - (A_i^* + 1) + 1 = k - (A_i^* + A_j^*)$.

Now by (9′) and (10), each of the q parameters $a(0)$, $a(p), \cdots, a((q-1)p)$ is subject to the same restrictions; so for a given set of auxiliary parameters $A_1, A_2, \cdots, A_{p-1}$, with $|A_x| \leqq k - 1$, there is either no corresponding semiregular n-gon with $1 \leqq a(i) \leqq k$, when $A_i^* + A_j^* \geqq k$, or the number of such n-gons is given by $[k - (A_i^* + A_j^*)]^q$. Hence we obtain

$$T(pq, k) = \Sigma[k - (A_i^* + A_j^*)]^q, \tag{12}$$

summed over all sets $A_1, A_2, \cdots, A_{p-1}$ having $A_i^* + A_j^* < k$.

If we let $h_p(B)$ count the number of sets $A_1, A_2, \cdots, A_{p-1}$ which have $B = A_i^* + A_j^*$, then we obtain the simpler form

$$T(pq, k) = \sum_{B=0}^{k-1} h_p(B)(k - B)^q. \tag{12′}$$

For any integer $t \geqq 2$, we consider a lattice point P in $t - 1$ dimensional Euclidean space E^{t-1}. If P has coordinates $A_1, A_2, \cdots, A_{t-1}$, we define $B(P) = A_i^* + A_j^*$, as in the discussion above. Let $h_t(B)$ count the number of lattice points P in E^{t-1} having $B(P) = B$. In evaluating (12′) we need the case $t = p$; but for later use we need $h_t(B)$ when t is not prime.

Clearly, $h_t(0) = 1$, for $B(P) = 0$ if and only if every $A_x = 0$.

To determine $h_t(B)$ when $B > 0$, we examine all possible cases, as follows:

(a) The points P which have all coordinates non-negative, $0 \leqq A_x \leqq B$, number $(B+1)^{t-1}$; those which have all A_x in the range $0 \leqq A_x \leqq B - 1$ number B^{t-1}; hence those which have at least one coordinate equal to B, and thus belong to the set counted by $h_t(B)$, are in number $(B+1)^{t-1} - B^{t-1}$.

(b) Similarly, the points P which have all coordinates non-positive and which belong to the set counted by $h_t(B)$ by having at least one coordinate equal to $-B$, are in number $(B+1)^{t-1} - B^{t-1}$.

(c) To count the cases where all coordinates A_x of P are in the range $-s \leqq A_x \leqq B-s$, with both $-s$ and $B-s$ occurring at least once each, so that $B(P) = A_i^* + A_j^* = (B-s) + |-s| = B$, we will use

$$(B+1)^{t-1} - B^{t-1} - B^{t-1} + (B-1)^{t-1};$$

for the terms used count, respectively, the entire population, those without $-s$, those without $B-s$, and then, to correct properly, those with neither $-s$ nor $B-s$ occurring.

Since s may range from 1 to $B-1$, there are $B-1$ cases like (c). Combining (a), (b) and (c), we have

$$h_t(B) = 2[(B+1)^{t-1} - B^{t-1}] + (B-1)[(B+1)^{t-1} - 2B^{t-1} + (B-1)^{t-1}]$$

which simplifies to

$$(13) \qquad h_t(B) = (B+1)^t - 2B^t + (B-1)^t, \qquad t \geqq 2, B \geqq 1.$$

For example, using $h_2(0) = 1$ and $h_2(B) = 2$ when $B \geqq 1$, we have by (12′)

$$T(2q,k) = k^q + 2 \sum_{B=1}^{k-1} (k-B)^q.$$

This simplifies to

$$(14) \qquad T(2q,k) = \sum_{x=1}^{k} x^q + \sum_{x=1}^{k-1} x^q,$$

involving sums of powers of integers for which there are well-known formulas. (See [1].)

For another example, since $h_3(0) = 1$ and by (13) we have $h_3(B) = 6B$ when $B \geqq 1$, we use (12′) and obtain

$$(15) \qquad \begin{aligned} T(3q,k) &= k^q + 6 \sum_{B=1}^{k-1} B(k-B)^q \\ &= k^q + 6 \sum_{r=0}^{q} (-1)^r k^{q-r} \binom{q}{r} \sum_{B=1}^{k-1} B^{r+1}. \end{aligned}$$

In particular, we find

$$(14') \qquad T(6,k) = k^2(k^2+1)/2;$$

$$(15') \qquad T(15,k) = k(2k^6 + 7k^4 + 7k^2 - 2)/14.$$

Let R_d be a rotation of G of period d, $d > 1$, where d divides pq. We know there are $\phi(d)$ rotations of period d, each having the same value for $N(R_d)$. We shall show $N(R_p) = k^q$, $N(R_q) = k^p$, $N(R_{pq}) = k$.

It is obvious that $N(R_{pq}) = k$, since only the regular n-gons, with $a(v) = a(v+1)$ for every v, are invariant under a rotation of period n.

A semi-regular polygon satisfying (9′) and invariant under R_p must satisfy the additional condition $a(0) = a(xq)$ for all x. But this requires $A_x = 0$ for all x. Conversely, $A_x = 0$ for all x implies $a(v) = a(v+q)$ for all v. Consequently, the free parameters in (9′) are $a(yp)$, $0 \leqq y \leqq q-1$. Under (10) each of these has k possible values, so $N(R_p) = k^q$.

We interchange the roles of p and q to obtain $N(R_q) = k^p$.

The Subcase $p = 2 < q$. Let D' be a reflection of G which leaves no edge fixed. Let the notation be chosen so that D' interchanges $a(i)$ and $a(2q - i + 1)$ for all i. In (9′) we have $1 \leqq x \leqq p - 1$, so there is only one auxiliary parameter $A = A_1$. To (9′), written in the form

$$a(2y + q) = a(2y) - A, \qquad 0 \leqq y \leqq q - 1,$$

we adjoin

$$(D') \qquad a(i) = a(j) \text{ if } i + j \equiv 1 \bmod 2q.$$

(a) We can solve $(2y + q) + 2y \equiv 1 \bmod 2q$ for $\bar{y}$, unique mod q. Then from (D') and (9′) we have $a(2\bar{y}) = a(2\bar{y} + q) = a(2\bar{y}) - A$, which requires $A = 0$.

(b) We can solve $(2y_1 + q) + 2y_2 \equiv 1 \bmod 2q$, which reduces to $2(y_1 + y_2) \equiv 1 \bmod q$, to see that, except for $\bar{y}$, the other y's are distributed in pairs, such that from (D'), (9′) and (a): $a(2y_2) = a(2y_1 + q) = a(2y_1)$. So the independent parameters $a(2y)$ in (9′) are reduced in number to $(q-1)/2 + 1 = (q+1)/2$.

Conversely, the conditions arrived at in (a) and (b) are sufficient so that sides $a(i)$ satisfying (9′) also satisfy (D'). Hence $N(D') = k^{(q+1)/2}$. There are q reflections of type D'.

Let M' be a reflection of G which leaves two edges fixed. Let the notation be chosen so that M' interchanges $a(i)$ and $a(2q - i)$ for all i; but since subscripts are identified modulo $2q$, sides $a(q)$ and $a(2q)$ are left fixed. To (9′) in the form $a(2y + q) = a(2y) - A$, $0 \leqq y \leqq 2 - 1$, we adjoin

$$(M') \qquad a(i) = a(j) \text{ if } i + j \equiv 0 \bmod 2q.$$

(a) The condition $2y_1 + 2y_2 \equiv 0 \bmod 2q$ reduces to $y_1 + y_2 \equiv 0 \bmod q$, hence, in general, under (M') there are pairs of distinct y_1, y_2 for which it is required that $a(2y_1) = a(2y_2)$. The one exception, $y_1 = y_2 = \bar{y}$, occurs for $\bar{y} = q$, corresponding to $a(2\bar{y}) = a(2q)$. So, the independent parameters $a(2y)$ in (9′) are reduced in number to $(q-1)/2 + 1 = (q+1)/2$; and there is no special restriction on the auxiliary parameter A.

Conversely, the conditions arrived at in (a) are sufficient to make the polygon semi-regular and invariant under M'. By the same reasoning used in finding (12), we find

$$N(M') = \sum_{A=-k+1}^{k-1} (k - |A|)^{(q+1)/2} = \sum_{x=1}^{k} x^{(q+1)/2} + \sum_{x=1}^{k-1} x^{(q+1)/2}.$$

There are q reflections of type M'.

Combining the above results with (11) and (14), we find

$$S(2q,k) = \frac{1}{4q}\left\{\sum_{x=1}^{k} x^q + \sum_{x=1}^{k-1} x^q + k^q + (q-1)k^2 + (q-1)k + qk^{(q+1)/2} + q\left[\sum_{x=1}^{k} x^{(q+1)/2} + \sum_{x=1}^{k-1} x^{(q+1)/2}\right]\right\}$$

$$= \frac{1}{4q}\left[2\sum_{x=1}^{k} x^q + (q-1)(k^2+k) + 2q\sum_{x=1}^{k} x^{(q+1)/2}\right].$$

For example, after simplification, we find

$$S(6,\ k) = \binom{k+3}{4};$$

$$S(10,k) = k(k+1)(k^2+k+3)(k^2+k+4)/60.$$

The Subcase $3 \leqq p < q$. Each reflection F' of G fixes one edge. Let the notation be chosen so that F' interchanges $a(i)$ and $a(pq-i)$, if they are distinct, but by the same rule F' fixes $a(pq)$. To (9′) we adjoin

(F') $\qquad a(i) = a(j)$ if $i + j \equiv 0 \bmod pq.$

(a) It follows from (F') on the one hand that

$$a(y_1 p) = a(y_2 p) \text{ if } y_1 + y_2 \equiv 0 \bmod q.$$

(b) On the other hand, if $x_1 + x_2 \equiv 0 \bmod p$, then $a(x_1 q) = a(x_2 q)$ or $a(0) - A_{x_1} = a(0) - A_{x_2}$, hence

$$A_{x_1} = A_{x_2} \text{ if } x_1 + x_2 \equiv 0 \bmod p.$$

But the conditions obtained in (a) and (b) are sufficient for (F'). Indeed, let

$i \equiv y_1 p + x_1 q$, $j \equiv y_2 p + x_2 q$ mod pq. Then if $i + j \equiv (y_1 + y_2)p + (x_1 + x_2)q \equiv 0$ mod pq, we have $y_1 + y_2 \equiv 0$ mod q and $x_1 + x_2 \equiv 0$ mod p. Hence $a(i) = a(y_1 p) - A_{x_1} = a(y_2 p) - A_{x_2} = a(j)$.

Thus among the auxiliary A_x, $1 \leqq x \leqq p-1$, only half, for instance, the ones $1 \leqq x \leqq (p-1)/2$, remain independent; and among the parameters $a(yp)$, $0 \leqq y \leqq q-1$, only those with $0 \leqq y \leqq (q-1)/2$, that is, $(q+1)/2$ of them, remain independent. If we set $t - 1 = (p-1)/2$, then in using (13) we have $t = (p+1)/2$. By the same reasoning used in finding (12) and (12′), we obtain

$$N(F') = \sum_{B=0}^{k-1} h_{(p+1)/2}(B)(k-B)^{(q+1)/2}.$$

There are pq reflections of type F'.

Combining the results above by the use of (11), for $n = pq$, p and q distinct odd primes, we find

$$S(pq,k) = \left[\frac{1}{2pq}\sum_{B=0}^{k-1} h_p(B)(k-B)^q + (p-1)k^q + (q-1)k^p \right.$$
$$\left. + (p-1)(q-1)k + pq \sum_{B=0}^{k-1} h_{(p+1)/2}(B)(k-B)^{(q+1)/2}\right].$$

For example, consider the case $p = 3$, $q = 5$. Since $(p+1)/2 = 2$ and $(q+1)/2 = 3$, we can use (12′) to see that $N(F', 15) = T(6,k)$. It follows that

$$S(15,k) = \frac{1}{30}[T(15,k) + 2k^5 + 4k^3 + 8k + 15T(6,k)].$$

Using the previous computations in (14′) and (15′), we find

$$S(15,k) = k(k+1)(k+2)(2k^4 - 6k^3 + 49k^2 - 30k + 55)/420.$$

The Case $n = p^\alpha q^\beta$, $p \neq q$. The conditions

(3″) $a(v) - a(v + p^{\alpha-1}q^\beta) - a(v + p^\alpha q^{\beta-1}) + a(v + p^{\alpha-1}q^\beta + p^\alpha q^{\beta-1}) = 0,$

for every integer v, may be replaced by the equivalent conditions

(9″) $$a(s + yp^\alpha q^{\beta-1} + xp^{\alpha-1}q^\beta) = a(s + yp^\alpha q^{\beta-1}) - A_{s,x};$$
$$1 \leqq x \leqq p-1;\quad 0 \leqq y \leqq q-1;\quad 0 \leqq s \leqq p^{\alpha-1}q^{\beta-1} - 1.$$

The conditions (9″) are like the conditions (9′) for the case $n = pq$, but repeated $p^{\alpha-1}q^{\beta-1}$ times; hence

(12″) $$T(p^\alpha q^\beta, k) = [T(pq,k)]^{p^{\alpha-1}q^{\beta-1}},$$

and the results in (12) and (12′) complete the counting.

Let R_d be a rotation of G of period d, where d divides n and $d > 1$. There are $\phi(d)$ rotations of type R_d, and we can show that each has

$$N(R_d) = k^{n/d}.$$

To see this we note that a semi-regular polygon $\{a(i)\}$ of order $n = p^\alpha q^\beta$ is invariant under R_d if and only if

$$(R_d) \qquad a(i) = a(j) \quad \text{if } i \equiv j \bmod n/d.$$

But any $\{a(i)\}$ satisfying (R_d) will satisfy (9″) because each set $\{a(i + xn/d)\}$ with $a(i + xn/d) = a(i)$ for $0 \leqq x \leqq d - 1$ can be used to form a regular polygon P_i of order d; then the n/d regular polygons P_i, for i ranging from 1 to n/d, may be composed to form a semi-regular polygon of order n. Since the $a(i)$ are to satisfy (10), we find $N(R_d) = k^{n/d}$.

The Subcase $n = p^\alpha q^\beta$, $3 \leqq p < q$. For this subcase, n is odd, so there is only one type of reflection F'' in G, fixing, say, edge $a(0)$. Then $a(i)$ is invariant under F'' if and only if

$$(\mathrm{F}'') \qquad a(i) = a(j) \text{ if } i + j \equiv 0 \bmod n.$$

Consider the $p^{\alpha-1}q^{\beta-1}$ polygons P_s of type pq, represented by the conditions (9″) as s ranges from 0 to $p^{\alpha-1}q^{\beta-1} - 1$. Since $a(0)$ is fixed by F'', the polygon P_0 is invariant under F'' only if it satisfies the conditions used in counting $N(F')$ when $n = pq$. So, P_0 contributes a factor $N(F')$ to the present computation. The other polygons P_s, $s \neq 0$, are divided into pairs P_{s_1} and P_{s_2}, where P_{s_2} is a reflection of P_{s_1} under F''. Since P_{s_1} satisfies a set of conditions like (9′), there are $T(pq, k)$ choices for P_{s_1}, with P_{s_2} completely determined by P_{s_1}. Since there are $(p^{\alpha-1}q^{\beta-1} - 1)/2$ pairs of polygons, we find

$$N(F'') = N(F')[T(pq,k)]^{(p^{\alpha-1}q^{\beta-1}-1)/2}.$$

For this subcase the computation of $S(n,k)$ is completed by using (11). For $n = p^\alpha q^\beta$, $3 \leqq p < q$,

$$S(p^\alpha q^\beta, k) = \frac{1}{2\,p^\alpha q^\beta}\Big\{[T(pq,k)]^{p^{\alpha-1}q^{\beta-1}} + \sum_{\substack{d|n \\ d>1}} \phi(d)\,k^{n/d} + p^\alpha q^\beta N(F')[T(pq,k)]^{(p^{\alpha-1}q^{\beta-1}-1)/2}\Big\}.$$

The simplest illustration is $n = 45 = 3^2 \cdot 5$, for which

$$S(45,k) = \frac{1}{90}[T(15,k)^3 + 2k^{15} + 4k^9 + 6k^5 + 8k^3 + 24k$$
$$+ 45N(F',15)T(15,k)].$$

Since $N(F',15) = T(6,k)$ we can use (14′) and (15′) to obtain $S(45,k)$ as a polynomial in k.

When $n = 2^\alpha q^\beta$, we find it necessary to distinguish the cases $\alpha = 1$ and $\alpha \geqq 2$.

The Subcase $n = 2q^\beta, 2 < q$. The condition (9″) describes $q^{\beta-1}$, an odd number, of component semi-regular polygons P_s of type $2q$.

Let D'' be a reflection of G fixing no edge. Suppose $\{a(i)\}$ is a semi-regular n-gon, invariant under D''. One of the P_s, say P_{s_0}, is reflected into itself, allowing $N(D')$ possibilities. The other P_s, $s \neq s_0$, are reflected in pairs. One member of the pair is an arbitrary semi-regular $2q$-gon. Since there are $(q^{\beta-1} - 1)/2$ pairs, we find

$$N(D'') = N(D')[T(2q,k)]^{(q^{\beta-1}-1)/2}.$$

Let M'' be a reflection of G fixing two edges. Suppose $\{a(i)\}$ is a semi-regular n-gon, invariant under M''. One of the P_s, say P_{s^*}, is reflected into itself, allowing $N(M')$ possibilities. The other P_s, $s \neq s^*$, are reflected in pairs. One member of a pair is an arbitrary semi-regular $2q$-gon. Hence we find

$$N(M'') = N(M')[T(2q,k)]^{(q^{\beta-1}-1)/2}.$$

Using (11) we have, for every odd prime q,

$$S(2q^\beta,k) = \frac{1}{4q^\beta}\left\{[T(2q,k)]^{q^{\beta-1}} + \sum_{\substack{d|n \\ d>1}} \phi(d)k^{n/d} + q^\beta N(D'') + q^\beta N(M'')\right\}.$$

For example, for $n = 18 = 2 \cdot 3^2$, we have

$$S(18,k)$$
$$= \frac{1}{36}\left\{T(6,k)^3 + k^9 + 2k^6 + 2k^3 + 6k^2 + 6k + 9\left[k^2 + \frac{k(2k^2+1)}{3}\right]T(6,k)\right\}$$
$$= k(k+1)(k^{10} - k^9 + 4k^8 + 4k^7 - k^6 + 25k^5 + 28k^4 + 8k^3 + 28k^2 + 48)/288.$$

The Subcase $n = 2^\alpha q^\beta$, $\alpha \geqq 2$, $2 < q$. The conditions (9″), together with $\alpha \geqq 2$, describe $2^{\alpha-1}q^{\beta-1}$, an even number, of component semi-regular polygons P_s of type $2q$.

Let D'' be a reflection of G fixing no edge. Suppose $\{a(i)\}$ is a semi-regular n-gon, invariant under D''. Then the P_s are reflected in pairs. One member of each pair is an arbitrary semi-regular $2q$-gon. There are $2^{\alpha-2}q^{\beta-1}$ pairs, hence under (10) we find

$$N(D'') = [T(2q,k)]^{2^{\alpha-2}q^{\beta-1}}.$$

Let M'' be a reflection of G fixing two edges. Suppose $\{a(i)\}$ is a semi-regular n-gon, invariant under M''. Then one of the P_s, say P_{s_0}, is reflected into itself, as under D'; and another, say P_{s^*}, is reflected into itself, as under M'. For instance, if $s_0 = 0$ then $s^* = 2^{\alpha-2}q^{\beta-1}$. The other P_s, $s \neq s_0$, $s \neq s^*$, are reflected in pairs. One member of each pair is an arbitrary semi-regular $2q$-gon. There are $2^{\alpha-2}q^{\beta-1} - 1$ pairs, hence under (10) we find

$$N(M'') = N(D')N(M')[T(2q,k)]^{2^{\alpha-2}q^{\beta-1}-1}.$$

Using (11) we have, for every odd prime q and every $\alpha \geqq 2$,

$$S(2^\alpha q^\beta, k) = \frac{1}{2^{\alpha+1}\,q^\beta}\left\{[T(2q,k)]^{2^{\alpha-1}q^{\beta-1}} + \sum_{\substack{d|n \\ d>1}} \phi(d)\,k^{n/d} + 2^{\alpha-1}q^\beta[N(D'') + N(M'')]\right\}.$$

For example, for $n = 12 = 2^2 \cdot 3$, we have

$$\begin{aligned} S(12,k) &= \frac{1}{24}\left\{T(6,k)^2 + k^6 + 2k^4 + 2k^3 + 2k^2 + 4k + 6\left[T(6,k) + k^2 \cdot \frac{k(2k^2+1)}{3}\right]\right\} \\ &= k(k+1)(k^6 - k^5 + 7k^4 + 9k^3 + 12k^2 + 4k + 16)/96. \end{aligned}$$

Part 3. Problems and comments

For a general n we have no trouble finding conditions (3*) which embrace (3), (3′) and (3″).

THEOREM. *If the positive integer $n > 2$ has the form*

$$n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m},$$

where $p_1, p_2, \cdots, p_m$ are distinct primes, then necessary and sufficient for a set $a(1), a(2), \cdots, a(n)$ of positive integers to be the sides of a semi-regular polygon of order n is that

$$a(v) + \sum_{r=1}^{m}\left[\Sigma^*(-1)^r a\left(v + \sum_{j=1}^{r} n/p_{x_j}\right)\right] = 0 \qquad (3^*)$$

for every integer v, where it is understood that (1) *applies and that Σ^* is taken over* *all* $\binom{m}{r}$ *combinations of* $p_1, p_2, \cdots, p_m$, *taken r at a time.*

The proof of the Theorem parallels exactly the proofs of (3) and (3′), except that $Q(z) = G(z)P(z)$ is formed by taking

$$G(z) = \prod_{i=1}^{m} (z^{n/p_i} - 1).$$

Thus $G(z)$ has as a root every imprimitive n-th root of unity, but no primitive nth root of unity.

However, we are unable to interpret the conditions (3*), in order to count $T(n,k)$, expect in the cases $m = 1$ and $m = 2$. The difficulty seems to center around the fact that each condition in (3*) involves

$$1 + \sum_{r=1}^{m} \binom{m}{r} = 2^m$$

of the $a(i)$. Thus for $m = 3$ we have

$$a(i) - a(i + n/p_1) - a(i + n/p_2) - a(i + n/p_3) + a(i + n/p_2 + n/p_3)$$
$$+ a(i + n/p_3 + n/p_1) + a(i + n/p_1 + n/p_2) - a(i + n/p_1 + n/p_2 + n/p_3) = 0,$$

for every integer i.

Of course, we know that only $\phi(n)$ of the conditions in (3*) are independent; and the conditions may be obtained in independent and different-appearing form by using the cyclotomic polynomial $\Phi_n(z)$ to determine $P_1(z) \equiv P(z) \bmod \Phi_n(z)$, where $P_1(z)$ is of degree $\phi(m) - 1$. But even in a specific case, such as $n = 30$, we have been unable to determine $T(n,k)$ when $m \geqq 3$.

It is somewhat enlightening to consider semi-regular polygons over the real field, in contrast to the previously considered semi-regular polygons over the rational field. For $n \geqq 3$, the irreducible polynomial $F_n(z)$ over the reals, having $\eta_0 = e^{2\pi i/n}$ as a root, is a quadratic. But over the rationals, the irreducible cyclotomic polynomial $\Phi_n(z)$, having η_0 as a root, is of degree $\phi(n)$. For $n \geqq 3$, $\phi(n) = 2$ only for $n = 3, 4, 6$. So, in these cases and only these cases, the conditions for a real semi-regular polygon are the same as those for a rational semi-regular polygon. We thus obtain in these three cases the following statements of which the first two are evident, while the third one follows easily from elementary geometrical considerations:

All semi-regular triangles, real as well as rational, are necessarily equilateral.

All semi-regular quadrilaterals, real as well as rational, are necessarily rectangles.

All semi-regular hexagons, real as well as rational, must necessarily have $a(i) - a(i+2) - a(i+3) + a(i+5) = 0$ for all i.

In all other cases $n \geqq 3$, $n \neq 3, 4, 6$, the real semi-regular polygons have only two sides completely dependent on the other sides, whereas the rational semi-regular polygons have $\phi(n)$ sides completely dependent on the other sides; since $\phi(n) > 2$, the outcome is quite different. For example, if n is a prime p, the only rational semi-regular p-gons are the regular ones; but if $p > 3$, there are real semi-regular p-gons which are not regular.

It is not necessary to make an algebraic analysis of the real semi-regular polygons of order $n \geqq 3$. For one sees geometrically from a consideration of angles and half-lines that $n - 2$ sides of the polygon, except for some obvious restrictive inequalities, can be chosen quite arbitrarily.

To describe the situation in more detail, let us denote the vertices of the polygon in counter-clockwise order by $A_0, A_1, A_2, \cdots, A_{n-1}, A_n = A_0$; let the side $A_{j-1}A_j$ have length x_j, and let the direction from A_{j-1} to A_j be $2\pi j/n$, $j = 1, 2, \cdots, n$. We shall describe how the quantities $x_1, x_2, \cdots, x_n$ in this order should be chosen, in order that $A_0A_1A_2 \cdots A_{n-1}A_n$ be a real semi-regular polygon.

For $1 \leqq j \leqq n/2$, the x_j may be quite arbitrary, subject only to the restriction that $x_j > 0$. For an index j in the range $n/2 < j \leqq n - 2$, we assume that the lengths $x_1, x_2, \cdots, x_{j-1}$, hence also the points $A_0, A_1, A_2, \cdots, A_{j-1}$ have already been chosen and investigate the choice of x_j. It turns out that x_j is subject to certain limits, above as well as below, but that the nature of the lower limit depends on the position of A_{j-1}. The two cases are illustrated in Figs. 1 and 2. In both Figures we have drawn the half-lines A_0X and A_0Y; A_0X has direction $2\pi(j+1)/n - \pi$ which is opposite to the direction from A_j to A_{j+1}, the next side to be dealt with, and A_0Y has direction π which means it will eventually contain the side $A_{n-1}A_n$

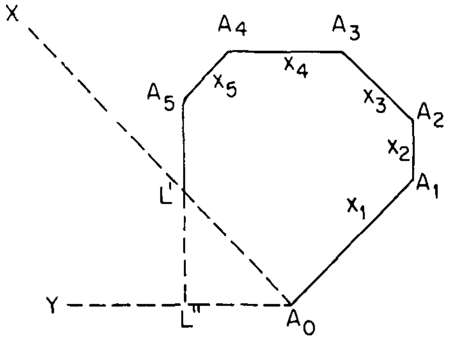

Figure 1

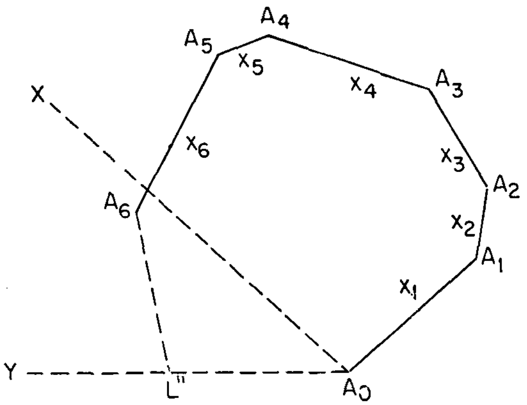

Figure 2

$= A_{n-1}A_0$. It is clear from the geometry of our configuration that A_j has to lie in the interior of the sector XA_0Y (by which we mean the sector to the "left" of A_0X and "above" A_0Y) in order that we be able to complete the semi-regular n-gon after the choice of x_j. Now in Fig. 1, where $n = 8, j = 6$, A_5 is still outside the sector XA_0Y, namely, to the right of A_0X; hence A_6 must be an interior point of the line segment $L'L''$, in symbols, $A_5L' < x_6 < A_5L''$. On the other hand, in Fig. 2, where $n = 9, j = 7$, A_6 is already inside the sector XA_0Y (or on A_0X); hence A_7 must be interior to the line segment A_6L'', in symbols, $0 < x_7 < A_6L''$. Finally, the sides x_{n-1} and x_n are completely determined by $x_1, x_2, \cdots, x_{n-2}$; in fact, as is easily seen, they are linear combinations of these $n-2$ quantities with coefficients which are, incidentally, real algebraic numbers.

As a final illustration, we give the complete set of inequalities and equations for the cases $n = 5$ and $n = 8$. They are easily derived by means of elementary trigonometry. The notation $(u)^+$ for real u is defined by $(u)^+ = \max(u, 0)$, and the quantity λ stands for $\lambda = (\sqrt{5} - 1)/2 = 2\cos(2\pi/5)$.

$n = 5$:
$$x_1 > 0;\ x_2 > 0;\ (\lambda x_1 - \lambda x_2)^+ < x_3 < (\lambda + 1)x_1 + x_2;$$
$$x_4 = x_1 + \lambda x_2 - \lambda x_3;\ x_5 = -\lambda x_1 + \lambda x_2 + x_3.$$

$n = 8$:
$$x_1 > 0; x_2 > 0; x_3 > 0; x_4 > 0;$$
$$(x_1 - x_3 - \sqrt{2}x_4)^+ < x_5 < x_1 + \sqrt{2}x_2 + x_3;$$
$$(\sqrt{2}x_1 + x_2 - x_4 - \sqrt{2}x_5)^+ < x_6 < x_1/\sqrt{2} + x_2 + x_3/\sqrt{2} - x_5/\sqrt{2};$$
$$x_7 = x_1 + \sqrt{2}x_2 + x_3 - x_5 - \sqrt{2}x_6;$$
$$x_8 = -\sqrt{2}x_1 - x_2 + x_4 + \sqrt{2}x_5 + x_6.$$

In contrast to the above relations, we recall that in the rational case the corresponding relations, when written in the same form, would be as follows: For $n = 5$, $a(1) > 0$, $a(2) = a(1)$, $a(3) = a(1)$, $a(4) = a(1)$, $a(5) = a(1)$; and for $n = 8$, $a(1) > 0$, $a(2) > 0$, $a(3) > 0$, $a(4) > 0$, $a(5) = a(1)$, $a(6) = a(2)$, $a(7) = a(3)$, $a(8) = a(4)$.

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